An invariant theoretic description of the primitive elements of the mod $-p$ cohomology of a finite loop space which are annihilated by Steenrod operations

Nondas E. Kechagias
Department of Mathematics, University of Ioannina, 45110, Greece

December 5, 2002

Abstract

We give an invariant theoretic description of the primitive elements in the mod $-p$ cohomology of a finite loop space $\Omega^{2n+1}S^{2n+1}$ for p odd. We also calculate the primitive elements which are annihilated by all Steenrod operations.

1 Introduction

In ([2]), the homology and cohomology of $\Omega^{m+1}S^{m+1}$ is described in terms of Dyer-Lashof operations for m finite or infinite. In ([10]) and ([2]), the dual of the Dyer-Lashof algebra was calculated for p even and odd respectively and its connection with modular invariants, namely the Dickson algebra, was known since then by the experts. That connection was explicitly formulated by Mui in ([5]) for $p = 2$ and later by Kechagias for p odd in ([6]). The advantage with modular invariants is that computations are easier in most cases because the Adem relations are overcome.

Inspired by work of Campbell, Peterson and Selick ([1]), we give an invariant theoretic description of the primitive elements in $H^*(\Omega_0^{2n+1}S^{2n+1};\mathbb{Z}/p\mathbb{Z})$, $PH^*(\Omega_0^{2n+1}S^{2n+1};\mathbb{Z}/p\mathbb{Z})$, and we use that description to calculate those primitive elements which are annihilated by all Steenrod operations for p odd, Theorem 41. That was a key and lengthy calculation in ([1]). In our approach the advantage is that computations are easier and the disadvantage that it requires some amount of preliminary work because we do not assume any familiarity with the connection between the Dyer-Lashof and Dickson algebras.

Since components of $\Omega^{m+1}S^{m+1}$ are homotopy equivalent, translation in homology is non-trivial namely, if $x \in H_*(\Omega^{m+1}S^{m+1};\mathbb{Z}/p\mathbb{Z})$ and $x_c \in H_0(\Omega^{m+1}S^{m+1};\mathbb{Z}/p\mathbb{Z})$ denotes the component of x then $x \cdot x_c^{-1} \in H_*(\Omega_0^{m+1}S^{m+1};\mathbb{Z}/p\mathbb{Z})$. Thus we will be considering the base point component only.

For each non-negative integer we define a subalgebra, $D[k] \otimes S(E_k)^{GL_k}$, of the ring of invariants $H^*((BZ/p\mathbb{Z})^k, \mathbb{Z}/p\mathbb{Z})^{GL_k}$ which was calculated by Mui [4]. That subalgebra turns out to be dual to the Dyer-Lashof coalgebra $R[k]$, [6]. We define a map j_n from $D[k] \otimes S(E_k)^{GL_k}$ to the quotient algebra of monomials with monomial degree at most n using the natural decomposition $\hat{\imath} : H^*((BZ/p\mathbb{Z})^k, \mathbb{Z}/p\mathbb{Z})^{GL_k} \hookrightarrow H^*((BZ/p\mathbb{Z})^k, \mathbb{Z}/p\mathbb{Z})^{B_k}$. Here B_k stands for the Borel subgroup. Let $j_n((d_{k,0}, M_{k;i}, M_{k;0,i})_{Ideal})$ be the image of the ideal $(d_{k,0}, M_{k;i}, M_{k;0,i})_{Ideal}$, then $\bigoplus_k j_n((d_{k,0}, M_{k;i}, M_{k;0,i})_{Ideal}) \cong PH^*(\Omega_0^{2n+1} S^{2n+1}; \mathbb{Z}/p\mathbb{Z})$, Theorem 45. Because the exterior generators of $(d_{k,0}, M_{k;i}, M_{k;0,i})_{Ideal}$ are not annihilated by all Steenrod operations, we compute $Ann j_n((d_{k,0})_{Ideal})$ using the Steenrod algebra action on Dickson generators and properties of the map j_n in section 3.

A would like to congratulate Eddy Campbell and David Wehlau for organizing that interesting and fertile conference. Finally, I would like to thank the referee very much for his valuable suggestions.

2 A principal ideal in the Dickson algebra

Let V^ℓ denote a $\mathbb{Z}/p\mathbb{Z}$-vector space with basis $\{e_1, ..., e_\ell\}$ for $1 \leq \ell \leq k$. Let Σ_{p^k} be the symmetric group which acts on V^k by permutations. V^k can also be considered as a subgroup of Σ_{p^k} acting by translations and we have a representation

$$\rho : W_{\Sigma_{p^k}}(V^k) \longrightarrow Aut(V^k) \equiv GL_k$$

V^k is also an elementary abelian p-subgroup and $H^*(BV^k, \mathbb{Z}/p\mathbb{Z}) \cong E(x_1, \cdots, x_k) \otimes P[y_1, \cdots, y_k]$ where $\{x_1, \cdots, x_k\}$ is a fixed basis for the dual of V^k. Here $|y_i| = 2 = 2|x_i|$ and $\beta(x_i) = y_i$, where $\beta(-)$ is the Bockstein operation. The contragradient representation of ρ induces an action of $Aut(V^k) \equiv GL_k$ on the graded algebra $E(x_1, \cdots, x_k) \otimes P[y_1, \cdots, y_k]$. Let $E_k := E(x_1, \cdots, x_k)$ and $S_k = P[y_1, \cdots, y_k]$.

The following theorems are well known:

Theorem 1 *[3] The Dickson algebra $S_k^{GL_k} := D[k] = \mathbb{Z}/p\mathbb{Z}[d_{k,0}, \cdots, d_{k,k-1}]$ is a polynomial algebra and degrees are $|d_{k,i}| = 2\left(p^k - p^i\right)$.*

Let B_k be the Borel subgroup of GL_k.

Theorem 2 *[4] $S_k^{B_k} := B[k] = \mathbb{Z}/p\mathbb{Z}[h_1, \cdots, h_k]$ is a polynomial algebra and degrees are $|h_i| = 2p^{i-1}(p-1)$.*

Note that for convenience we call h_i what in the literature stands for h_i^{p-1}.

Let $\hat{\imath} : D[k] \hookrightarrow B[k]$ be the natural inclusion. This map is described by the following relations.

Let $f_{\ell-1}(x) = \prod_{u\in V^{\ell-1}} (x-u)$, then $f_{\ell-1}(x) = \sum_{i=0}^{\ell-1} (-1)^{k-i} x^{p^i} d_{\ell-1,i}$ and $h_\ell = \prod_{u\in V^{\ell-1}} (y_\ell - u)^{p-1}$. Moreover, (see [6]),

$$\hat{\imath}(d_{k,k-i}) = \sum_{1\le j_1<\cdots<j_i\le k} \prod_{s=1}^{i} (h_{j_s})^{p^{k-i+s-j_s}} \tag{1}$$

Both $B[k]$ and $D[k]$ are algebras over the Steenrod algebra, A, and its action on generators has been given in [7].

Mui gave an invariant theoretic description of the cohomology algebra of a symmetric group and calculated rings of invariants involving the exterior algebra E_k as well in [4].

Theorem 3 *[4]1) The algebra $(E_k \otimes S_k)^{B_k}$ is a tensor product of the polynomial algebra $B[k]$ and the $\mathbb{Z}/p\mathbb{Z}$-module spanned by the set of elements consisting of the following monomials:*

$$M_{s;s_1,\ldots,s_m} L_s^{p-2};\ 1\le m\le k,\ m\le s\le k,\ \text{and } 0\le s_1<\cdots<s_m = s-1.$$

Its algebra structure is determined by the following relations:
a) $(M_{s;s_1} L_s^{p-2})^2 = 0$, for $1\le s\le k, 0\le s_1\le s-1$.
b) $M_{s;s_1,\ldots,s_m} L_s^{p-2} (L_s^{p-1})^{m-1} =$
$(-1)^{m(m-1)/2} \prod_{q=1}^{m} (\sum_{r=s_q+1}^{s} M_{r;r-1} L_r^{p-2} h_{r+1}\ldots h_s d_{r-1,s_q})$
Here $1\le m\le k$, $m\le s\le k$, and $0\le s_1<\cdots<s_m = s-1$.

2) The algebra $(E_k\otimes S_k)^{GL_k}$ is a tensor product of the polynomial algebra $D[k]$ and the $\mathbb{Z}/p\mathbb{Z}$-module spanned by the set of elements consisting of the following monomials:

$$M_{k;s_1,\ldots,s_m} L_k^{p-2};\ 1\le m\le k,\ \text{and } 0\le s_1<\cdots<s_m\le k-1.$$

Its algebra structure is determined by the following relations:
a) $(M_{k;s_1,\ldots,s_m} L_k^{p-2})^2 = 0$ for $1\le m\le k$, and $0\le s_1<\cdots<s_m\le k-1$.
b) $M_{k;s_1,\ldots,s_m} L_k^{(p-2)} d_{k,k-1}^{m-1} = (-1)^{m(m-1)/2} M_{k;s_1} L_k^{p-2} \ldots M_{k;s_m} L_k^{p-2}$.
Here $1\le m\le k$, and $0\le s_1<\cdots<s_m\le k-1$.

The elements $M_{k;s_1,\ldots,s_m}$ above have been defined by Mui in [4] as follows:

$$M_{k;s_1,\ldots,s_m} = \frac{1}{m!} \begin{vmatrix} x_1 & \cdots & x_1 \\ \vdots & & \vdots \\ x_1 & \cdots & x_k \\ y_1 & \cdots & y_k \\ \vdots & & \vdots \\ y_1^{p^{k-1}} & \cdots & y_k^{p^{k-1}} \end{vmatrix} \text{ and } L_k = \begin{vmatrix} y_1 & \cdots & x_k \\ y_1^p & \cdots & y_k^p \\ \vdots & & \vdots \\ y_1^{p^{k-1}} & \cdots & y_k^{p^{k-1}} \end{vmatrix}$$

Here there are m rows of x_i's and the s_i-th's powers are omitted, where $0 \leq s_1 < \cdots < s_m \leq k-1$ in the first determinant.

The degree of elements above are $|M_{k;s_1,\ldots,s_m}| = m + 2((1+\cdots+p^{k-1}) - (p^{s_1}+\cdots+p^{s_m}))$ and $|L_k^{p-2}| = 2(p-2)(1+\cdots+p^{k-1})$.

The rest of this section is devoted to the connection between the dual of the Dyer-Lashof, Dickson and the cohomology algebra of $\Omega^{2n+1}S^{2n+1}$ for p odd. For details between the Dyer-Lashof and Dickson algebras please see [8].

Definition 4 *Let $S(E_k)^{B_k}$ be the subspace of $(E_k \otimes S_k)^{B_k}$ generated by:*

$$M_{s;s-1}(L_s)^{p-2} \text{ for } 1 \leq s \leq k \ ,$$

$$\prod_{t=1}^{\ell} M_{s_{2t}+1;s_{2t-1},s_{2t}}(L_{s_{2t}+1})^{p-2} \text{ for } 0 \leq s_1 < \ldots < s_{2\ell} \leq k-1,$$

$$\prod_{t=1}^{\ell} M_{s_{2t}+1;s_{2t-1},s_{2t}}(L_{s_{2t}+1})^{p-2} M_{s;s-1}(L_s)^{p-2} \text{ for } 0 \leq s_1 < \ldots < s_{2\ell} < s \leq k$$

and $S(E_k)^{GL_k}$ be the subspace of $(E_k \otimes S_k)^{GL_k}$ generated by:

$$M_{k;s}(L_k)^{p-2} \text{ for } 0 \leq s \leq k-1 \ ,$$

$$\prod_{t=1}^{\ell} M_{k;s_{2t-1},s_{2t}}(L_k)^{p-2} \text{ for } 0 \leq s_1 < \ldots < s_{2\ell} \leq k-1,$$

$$M_{k;s-1}(L_k)^{p-2} \prod_{t=1}^{\ell} M_{k;s_{2t-1},s_{2t}}(L_k)^{p-2} \text{ for } 0 \leq s < s_1 < \ldots < s_{2\ell} < k$$

The map $\hat{\imath} : D[k] \hookrightarrow B[k]$ defined in 1 is extended to $\hat{\imath} : D[k] \otimes S(E_k)^{GL_k} \hookrightarrow B[k] \otimes S(E_k)^{B_k}$ by the following relations:

Lemma 5 *1)* $M_{k;s}L_k^{p-2} = M_{s+1;s}L_{s+1}^{p-2}h_{s+1}\ldots h_k + \sum_{t=2}^{k-s} M_{s+t;s+t-1}L_{s+t}^{p-2}d_{s+t-1,s}h_{s+t+1}\ldots h_k$.

2) $M_{k;s,m}L_k^{p-2} = M_{m+1;s,m}L_{m+1}^{p-2}h_{m+2}\ldots h_k + \sum_{t=2}^{k-m} M_{m+t;s,m+t-1}L_{m+t}^{p-2}d_{m+t-1,m}h_{m+t+1}\ldots h_k - M_{m+t;m,m+t-1}L_{m+t}^{p-2}d_{m+t-1,s}h_{m+t+1}\ldots h_k$.

Proof. We use induction on Mui's formula:
$M_{k-1;s_1,\ldots,s_m}L_{k-1}^{p-2}h_k = M_{k;s_1,\ldots,s_m}L_k^{p-2} - \sum_1^m (-1)^{m+i} M_{k;s_1,\ldots,\widehat{s_i},\ldots,k-1}d_{k-1,s_i}$.
■

Proposition 6 *1) $S(E_k)^{GL_k}$ is not closed under the Steenrod algebra action.*
2) $D[k] \otimes S(E_k)^{GL_k}$ is closed under the Steenrod algebra action.

Proof. This follows from Theorem 14 in [7]. ■

Let $D := \prod_{k \geq 0} D[k] \otimes S(E_k)^{GL_k}$ be the induced graded algebra where its identity element is $\prod_k ({}_k1)$. The height of elements in $D[k] \otimes S(E_k)^{GL_k}$ is defined to be k. The pair degree and height defines uniquely each element of D. The

unit in $D[k] \otimes S(E_k)^{GL_k}$ is of height k and it is not related with units with different height. D is not of finite type but D_+ is. We shall note that if we let $B := \prod_{k\geq 0} B[k] \otimes S(E_k)^{B_k}$, then B_+ is not of finite type because h_t appears with many different heights.

The augmentation $\epsilon : D \to \mathbb{Z}/p\mathbb{Z}$is given by $\epsilon(\prod_k \lambda_k({}_k1)) = \lambda_0$.

Next we define a coproduct in D and B as follows (see [6] page 277):

$$\psi : D[k] \otimes S(E_k)^{GL_k} \to \sum_{k\geq t\geq 0} (D[t] \otimes S(E_t)^{GL_t}) \bigotimes (D[k-t] \otimes S(E_{k-t})^{GL_{k-t}})$$

Definition 7 *i)* $\psi({}_k1) = \sum_t 1 \otimes_{k-t} 1$;
ii) $\psi(d_{k,i}) = \sum_{(t,j)} d_{t,0}^{p^{k-t}-p^j} d_{t,i-j}^{p^j} \otimes d_{k-t,j}$;
iii) $\psi(\prod d_{k,i}^{m_i}) = \sum_{(t,j_0,\ldots,j_{k-1})} \prod_i \left(d_{t,0}^{m_i(p^{k-t}-p^{j_i})} d_{t,i-j_i}^{m_i p^{j_i}} \right) \otimes \prod_i d_{k-t,j_i}^{m_i}$;
iv) $\psi(M_{k;s}L_k^{(p-2)}) = \sum_{(t,j)} d_{t,0}^{p^{k-t}-p^j} d_{t,s-j}^{p^j} \otimes M_{k-t;j}L_{k-t}^{(p-2)} +$
$\sum_{(t)} d_{t,0}^{p^{k-t}-1} M_{t;s}L_t^{(p-2)} \otimes d_{k-t,0}$;
v) $\psi(M_{k;s,i}L_k^{(p-2)}) =$
$\sum_{(t,j,f)} d_{t,0}^{p^{k-t}-p^f-p^j} (d_{t,t+j-i}^{p^j} d_{t,t+f-s}^{p^f} - d_{t,t+f-i}^{p^f} d_{t,t+j-s}^{p^j}) \otimes M_{k-t;f,j}L_{k-t}^{(p-2)} -$
$\sum_{(t,j)} d_{t,0}^{p^{k-t}-p^j-1} (d_{t,t+j-i}^{p^j} M_{t;t-s}L_t^{(p-2)} - d_{t,t+j-s}^{p^j} M_{t;t-i}L_t^{(p-2)}) \otimes M_{k-t;j}L_{k-t}^{(p-2)} +$
$\sum_{(t)} d_{t,0}^{p^{k-t}-1} M_{t;t-s,t-i}L_t^{(p-2)} \otimes d_{k-t,0}$;
Moreover, the following rule is applied. $\left(\sum_t \sum_{l,l'} a_{t,l} \otimes a_{k-t,l'}\right) \left(\sum_t \sum_{q,q'} b_{t,q} \otimes b_{k-t,q'}\right) :=$
$\left(\sum_t \sum_{l,l',q,q'} a_{t,l} b_{t,q} \otimes a_{k-t,l'} b_{k-t,q'}\right)$.
vi) $\psi({}_kh_t) = \sum_{1\leq i\leq t-1} ({}_ih_i)^{p^{t-1-i}(p-1)} \otimes ({}_{k-i}h_{t-i}) + \sum_{0\leq i\leq k-t} ({}_{t+i}h_t) \otimes 1_{k-t-i}$;
vii) $\psi({}_mM_{k;k-1}L_k^{(p-2)}) = \sum_{1\leq i\leq k-1} h_i^{p^{k-1-i}(p-1)} \otimes_{m-i} (M_{k-i;k-i-1}L_{k-i}^{(p-2)}) +$
$\sum_{0\leq i\leq m-k} ({}_{k+i}M_{k;k-1}L_k^{(p-2)}) \otimes 1_{m-k-i}$.

Here ${}_kh_t$denotes that it is an element of $B[k]$ i.e. of height k. Now, D_+ is an algebra and a coalgebra but not a Hopf algebra because its coproduct on its unit $\prod_k({}_k1)$ is not well defined although it is well defined on $({}_k1)$. The same is true for B.

Let $d_{k,o}^{m_0} \ldots d_{k,k-1}^{m_{k-1}}$ be a typical element of $D[k]$ and $h_1^{j_1} \ldots h_k^{j_k}$ of $B[k]$. We abbreviate those elements by d_k^m and h_k^j respectively, where $m = (m_0, \ldots, m_{k-1})$ and $j = (j_1, \ldots, j_k)$ are elements of $\mathbb{N}^k$ the submonoid of $\mathbb{Z}^k$. Here $\mathbb{N}$stands for the set of non-negative integers and $\mathbb{Z}$for the integers. Let us also give a left

lexicographical ordering in those sequences and call j **admissible**, if $j_t \leq j_{t+1}$ for $1 \leq t \leq k-1$.

We extend the notion of sequences above to include elements of $S(E_k)^{GL_k}$ and $S(E_k)^{B_k}$. We consider $(\mathbb{N}, +)$ as a sub-monoid of $(\mathbb{Q}, +)$ and let $<\mathbb{N}, \frac{1}{2}>$ be the sub-monoid generated by $\mathbb{N}$and $\frac{1}{2}$ in $\mathbb{Q}$. Let $<\mathbb{N}, \frac{1}{2}>^k$ be the sub-monoid which is the k-th Cartesian product of $<\mathbb{Q}, \frac{1}{2}>$.

Definition 8 *Let $<\mathbb{N}, \frac{1}{2}>^k \times (\mathbb{Z}/2Z)^k$ be the Cartesian product of the sub-monoid $<\mathbb{N}, \frac{1}{2}>^k$ and the group $(\mathbb{Z}/2\mathbb{Z})^k$. Let $I = (i_1, ..., i_k)$ and $\varepsilon = (e_1, ..., e_k)$, then $(I, \varepsilon) \in <\mathbb{N}, \frac{1}{2}>^k \times (\mathbb{Z}/2\mathbb{Z})^k$ will be called admissible, if $0 \leq 2i_t - 2i_{t-1} + e_{t-1}$ for $2 \leq t \leq k-1$.*

Note that we will be comparing sequences corresponding to elements of the same degree.

Definition 9 *If $\mathring{A}$is a $\mathbb{Z}/p\mathbb{Z}$algebra, then we define $ß(\mathring{A})$ to be the monomial basis of $\mathring{A}$.*

Thus $ß(D[k] \otimes S(E_k)^{GL_k})$ and $ß(B[k] \otimes S(E_k)^{B_k})$ denote the vector space bases of monomials in $D[k] \otimes S(E_k)^{GL_k}$ and $B[k] \otimes S(E_k)^{B_k}$ respectively.

Definition 10 *Let $\chi_{\min}$ and $\chi_{\max}$ be the set functions from $ß(D[k] \otimes S(E_k)^{GL_k})$ ($ß(B[k] \otimes S(E_k)^{B_k})$) to the monoid $<\mathbb{N}, \frac{1}{2}>^k \times (\mathbb{Z}/2Z)^k$ given by*

$$1)\ \chi_{\min}(d_{k,i}) = (\underbrace{0, ..., 0}_{i}, \underbrace{1, ..., 1}_{k-i})x(0, ..., 0) \text{ and}$$

$$\chi_{\max}(d_{k,i}) = (p^{k-1}, ..., p^{k-i}, 0, ..., 0)x(0, ..., 0).$$

$$2)\ \chi_{\min}(M_{k;s}L_k^{(p-2)}) = (\underbrace{\frac{1}{2}, ..., \frac{1}{2}}_{s}, \underbrace{1, ..., 1}_{k-s})x(\underbrace{0, ..., 0}_{s}, 1, \underbrace{0, ..., 0}_{k-s-1}) \text{ and}$$

$$\chi_{\max}(M_{k;s}L_k^{(p-2)}) = (\underbrace{\frac{1}{2}, ..., \frac{1}{2}}_{s}, \underbrace{1\frac{1}{2}, ..., 1\frac{1}{2}}_{k-s-1}, 1)x(0, ..., 0, 1).$$

$$3)\ \chi_{\min}(M_{k;s,m}L_k^{(p-2)}) = (\underbrace{0, ..., 0}_{s}, \underbrace{\frac{1}{2}, ..., \frac{1}{2}}_{m-s}, \underbrace{1, ..., 1}_{k-m})x(\underbrace{0, ..., 0}_{s}, \underbrace{1, 0, ..., 0, 1}_{m-s}, \underbrace{0, ..., 0}_{k-m}) \text{ and}$$

$$\chi_{\max}(M_{k;s,m}L_k^{(p-2)}) = (\underbrace{0, ..., 0}_{m}, \underbrace{1\frac{1}{2}, ..., 1\frac{1}{2}}_{k-m-1}, 1)x(\underbrace{0, ..., 0}_{m}, \underbrace{1, 0, ..., 0, 1}_{k-m}).$$

and the rule $\chi_{\min}(dd'MM') = \chi_{\min}(d) + \chi_{\min}(d') + \chi_{\min}(M) + \chi_{\min}(M')$. Here d, $d' \in ß(D[k])$ and M, $M' \in ß(S(E_k)^{GL_k})$. The same holds for $\chi_{\max}$.

Note that the function $\chi_{\min}$ is always admissible and $\hat{\imath}(d_{k,i})$ contains a monomial with a unique admissible sequence, namely $h^{\chi_{\min}(d_{k,i})}$, and a monomial with a unique maximal sequence, namely $h^{\chi_{\max}(d_{k,i})}$. The same is true for elements

$M_{k;s-1}L_k^{p-2}$ and $M_{k;s,m}L_k^{p-2}$. Moreover, $\hat{\imath}(d_k^m M)$ might contain a number of monomials with admissible sequences and this is the main point of investigation because of its applications.

The function above define a natural function θ^*:

$$\theta^* : ß(D[k] \otimes S(E_k)^{GL_k}) \rightarrowtail ß(B[k] \otimes S(E_k)^{B_k}) \tag{2}$$

defined by $\theta^*(d^m) = h^{\chi_{\min}(d_k^m)}$, where $\chi_{\min}(d_k^m) = (i_1, ..., i_n)$ and $i_1 = m_0$, $i_t = m_0 + ... + m_{t-1}$. $\theta^*(M_{k;s}L_k^{(p-2)}) = M_{s+1;s}L_{s+1}^{(p-2)}h_{s+2}...h_k$ and $\theta^*(M_{k;s,m}L_k^{(p-2)}) = M_{m+1;s,m}L_{m+1}^{(p-2)}h_{m+2}...h_k$. Finally, $\theta^*(dM) = \theta^*(d)\theta^*(M)$.

Definition 11 *Let $R^*[k]$ be the vector space spanned by elements $(Q_{I,\varepsilon})^*$ where $(I, \varepsilon) = \chi_{\min}(d_k^m M)$ for all $d_k^m M$ in $ß(D[k] \otimes S(E_k)^{GL_k})$.*

We define the degree of $(Q_{I,\varepsilon})^*$ or by abuse of notation of the sequence (I, ε) such that the maps θ^*, $\chi_{\min}$ and $\chi_{\max}$ are degree preserving maps.

Definition 12 *Let $I = (i_1, ..., i_k)$ and $\varepsilon = (e_1, ..., e_k)$, then the degree of $(Q_{I,\varepsilon})^*$ or (I, ε) is $2(p-1)\left(\sum_{t=1}^{k} i_t p^{t-1}\right) - \left(\sum_{t=1}^{k} e_t p^{t-1}\right)$.*

Next we define an order on $<\mathbb{N}, \frac{1}{2}>^k \times (\mathbb{Z}/2\mathbb{Z})^k$ which is compatible with the order defined in$<\mathbb{N}, \frac{1}{2}>^k \equiv <\mathbb{N}, \frac{1}{2}>^k \times (0, ..., 0)$.

Definition 13 *Let (I, ε) be a typical element of $<\mathbb{N}, \frac{1}{2}>^k \times (\mathbb{Z}/2\mathbb{Z})^k$ and we call $I_t = (i_t, ..., i_k)$ and $\varepsilon_t = (e_t, ..., e_k)$ for $t = 1, ..., k$. We define $(I, \varepsilon) \geq (I', \varepsilon')$, if $2(p-1)i_t - e_t - |I_{t-1}, \varepsilon_{t-1}| \geq 2(p-1)i'_t - e'_t - |I'_{t-1}, \varepsilon'_{t-1}|$ for the smallest t.*

Let the vector space isomorphism $\Phi_k : D[k] \otimes S(E_k)^{GL_k} \to R^*[k]$ be given by

$$\Phi_k(d_k^m M) = \sum_{(I,\varepsilon) \geq \chi_{\min}(d_k^m M)} a_{m,(I,\varepsilon)}(Q_I)^*$$

Here $a_{m,(I,\varepsilon)}$ is the coefficient of h^I in $\hat{\imath}(d_k^m M)$ and all sequences (I, ε) are admissible.

Remark 14 *The map above is upper triangular with one along the main diagonal and it has been studied along with its inverse in [8] Theorem 45.*

Example 15 *Let us consider $\Phi_k(d_{3,0}d_{3,1}^{12}d_{3,2}^4)$ for $p = 3$.*

Using MAPLE we get $\hat{\imath}(d_{3,1}^{12}d_{3,2}^4) = 2h_1^{27}h_2^{15}h_3^{12} + h_1^{45}h_2^{27}h_3^6 + 2h_1^{45}h_2^{18}h_3^9 + 2h_1^{18}h_2^{18}h_3^{12} + h_1^{36}h_2^{27}h_3^7 + h_1^{27}h_2^{30}h_3^7 + 2h_1^{36}h_2^{18}h_3^{10} + 2h_1^{36}h_2^{21}h_3^9 + 2h_1^{36}h_2^{36}h_3^4 + 2h_1^{36}h_2^{12}h_3^{12} + h_1^{27}h_2^{33}h_3^6 + 2h_1^{36}h_2^{30}h_3^6 + h_1^{18}h_2^{27}h_3^9 + h_1^{45}h_3^{15} + h_2^{15}h_3^{15} + h_1^{72}h_2^{36} + h_1^{72}h_3^{12} + h_2^{39}h_1^{63} + h_2^{48}h_1^{36} + h_2^{24}h_3^{12} + h_1^{63}h_3^{13} + h_1^{36}h_3^{16} + h_2^{21}h_3^{13} + h_2^{12}h_3^{16} + 2h_1^9h_3^{13}h_2^{18} + 2h_1^{36}h_3^{15}h_2^3 + 2h_2^{12}h_3^{15}h_1^9 + h_1^{27}h_3^{15}h_2^6 + h_1^{18}h_3^{15}h_2^9 + h_1^{45}h_2^{45} + 2h_1^{45}h_2^{36}h_3^3 + 2h_2^{12}h_3^{13}h_1^{27} + 2h_1^{63}h_3^3h_2^{30} + h_2^{36}h_1^{63}h_3 + h_2^{30}h_1^9h_3^9 + h_2^3h_3^{13}h_1^{54} + h_1^{72}h_2^9h_3^9 + h_2^{42}h_3^3h_1^{27} + h_2^6h_3^{12}h_1^{54} + h_2^{33}h_3^3h_1^{54} + h_2^3h_3^{16}h_1^{27} + h_2^{27}h_1^9h_3^{10} + h_2^{30}h_3^4h_1^{54} + h_2^{45}h_1^{36}h_3 + 2h_1^{63}h_3^{12}h_2^3 + h_1^{72}h_3^3h_2^{27} + h_1^{63}h_3^4h_2^{27} + h_2^{39}h_3^4h_1^{27} + h_2^{12}h_1^{63}h_3^9 + h_2^9h_1^{63}h_3^{10} + h_1^9h_3^{16}h_2^9$. *The admissible sequences for $\Phi_k(d_{3,0}d_{3,1}^{12}d_{3,2}^4)$ are: $h^{1,13,17}$, $h^{1,16,16}$, $2h^{10,13,16}$ and $h^{10,10,17}$. Thus* $\Phi_k(d_{3,0}d_{3,1}^{12}d_{3,2}^4) = (Q_{1,13,17})^* + (Q_{1,16,16})^* + 2(Q_{10,13,16})^* + (Q_{10,10,17})^*$.

Of course, $R^*[k]$ inherits a Steenrod algebra structure from the isomorphism above (see [6]).

Let $\kappa = [\frac{k+1}{2}]$ and $\varepsilon = (e_1, ..., e_k)$, then $S(E_k)^{GL_k}$ is spanned by at most κ monomials:

$$M^\varepsilon := \begin{cases} M_{k;s_1,s_2}^{[\frac{e_1+e_2}{2}]} L_k^{p-2} ... M_{k;s_{k-1},s_k}^{[\frac{e_{k-1}+e_k}{2}]} L_k^{p-2}, \text{ if } k \text{ is even} \\ M_{k;s_1}^{e_1} L_k^{p-2} M_{k;s_2,s_3}^{[\frac{e_2+e_3}{2}]} L_k^{p-2} ... M_{k;s_{k-1},s_k}^{[\frac{e_{\kappa-1}+e_\kappa}{2}]} L_k^{p-2}, \text{ if } k \text{ is odd} \end{cases}$$

Definition 16 *Let*
$\sum(m,\varepsilon) := \begin{cases} m_0 + ... + m_{k-1} + [\frac{e_1+e_2}{2}] + ... + [\frac{e_{k-1}+e_k}{2}], \text{ if } k \text{ is even} \\ m_0 + ... + m_{k-1} + e_1 + [\frac{e_2+e_3}{2}] + ... + [\frac{e_{k-1}+e_k}{2}], \text{ if } k \text{ is odd} \end{cases}$ *and we call the monomial degree of $d_k^m M^\varepsilon$ to be the non-negative integer $\sum(m,\varepsilon)$.*

Let $\left(D[k] \otimes S(E_k)^{GL_k}\right)_n$ be the vector space spanned by all monomials with monomial degree n in $D[k] \otimes S(E_k)^{GL_k}$.

Let $R_n^*[k]$ be the vector space spanned by elements $(Q_{I,\varepsilon})^*$ where $(I,\varepsilon) = \chi_{\min}(d_k^m M^\varepsilon)$ for all $d_k^m M^\varepsilon$ in $\beta(D[k] \otimes S(E_k)^{GL_k})_n$.

Since we are considering homogeneous elements the monomial degree can be extended to polynomial degree as well.

Lemma 17 *$(Q_{I,\varepsilon})^* \in R_n^*[k]$ if and only if (I,ε) is admissible and $i_k \leq n$.*

Remark 18 *1) We should note that if $m > m'$ and $|d_k^m M^\varepsilon| = |d_k^{m'} M^{\varepsilon'}|$, then the polynomial degree of $d_k^{m'} M^{\varepsilon'}$ is greater of that of $d_k^m M^\varepsilon$.*

2) We also note that if $\Phi_k^{-1}(Q_I)^* = \sum_{I \leq \chi_{\min}(d_k^m)} a'_{m,I}(d_k^m M^\varepsilon)$, then the polynomial degree of $d_k^m M^\varepsilon$ is greater or equal than of i_k.

We shall make $(D[k] \otimes S(E_k)^{GL_k})_n$ a quotient subalgebra of $D[k] \otimes S(E_k)^{GL_k}$ and write its elements for the natural images for convenience.

Let the epimorphism $\pi_n : R^*[k] \to R_n^*[k]$ be given by

$$\pi_n(Q_I)^* = \begin{cases} (Q_I)^*, \text{ if } (Q_I)^* \in R_n^*[k] \\ 0, \text{ otherwise} \end{cases}$$

Lemma 19 *The epimorphism $\pi_n : R^*[k] \to R_n^*[k]$ is a map of algebras.*

Proof. We must show $\pi_n((Q_I)^*(Q_J)^*) = \pi_n(Q_I)^* \pi_n(Q_J)^*$. But

$$\pi_n(Q_I)^* \pi_n(Q_J)^* = \begin{cases} (Q_I)^*(Q_J)^*, \text{ if } i_k,\, j_k \leq n \\ 0, \text{ otherwise} \end{cases}$$

On the other hand, $\pi_n((Q_I)^*(Q_J)^*) = \pi_n \Phi_k(\Phi_k^{-1}(Q_I)^* \cdot \Phi_k^{-1}(Q_J)^*)$. Using remark 18 2), we get that both sides are equal. ■

Definition 20 *Let $j_n : (D[k] \otimes S(E_k)^{GL_k}) \to (D[k] \otimes S(E_k)^{GL_k})_n$ be the epimorphism defined by*

$$j_n(d_k^m M^\varepsilon) = \Phi_k^{-1}(\pi_n(\Phi_k(d_k^m M^\varepsilon)))$$

The map j_n is well defined because of remarks 14 and 18 and also an algebra epimorphism.

Example 21 *Let us continue on example 15 for $n = 16$. We recall that $\Phi_k(d_{3,0}d_{3,1}^{12}d_{3,2}^4) = (Q_{1,13,17})^* + (Q_{1,16,16})^* + 2(Q_{10,13,16})^* + (Q_{10,10,17})^*$. Since $n = 16$, $\pi_n(\Phi_k(d_{3,0}d_{3,1}^{12}d_{3,2}^4)) = (Q_{1,16,16})^* + 2(Q_{10,13,16})^*$. Because $2h^{10,13,16}$ is an element of $\hat{\imath}((\theta^*)^{-1}h^{1,16,16})$, $\Phi_k^{-1}(\pi_n(\Phi_k(d_{3,0}d_{3,1}^{12}d_{3,2}^4))) = d_{3,0}d_{3,1}^{15}$.*

Definition 22 *Let $(d_{k,0}, M_{k;i}, M_{k;0,i})_{Ideal}$ stand for the ideal of $D[k]\otimes S(E_k)^{GL_k}$ generated by ,$\{d_{k,0}, M_{k;i}, M_{k;0,i}\}$ and $j_n((d_{k,0}, M_{k;i}, M_{k;0,i})_{Ideal})$ its image in $(D[k]\otimes S(E_k)^{GL_k})_n$. Respectively, $j_n((d_{k,0})_{Ideal})$ stands for the image of the principal ideal $(d_{k,0})_{Ideal}$ in $D[k]_n$.*

Remark 23 *1) Because D_+ is of finite type,*
$\prod_k (d_{k,0}, M_{k;i}, M_{k;0,i})_{Ideal} \equiv \bigoplus_k (d_{k,0}, M_{k;i}, M_{k;0,i})_{Ideal}$.
2) It is immediate from the definition of the coproduct in D that $\bigoplus_k (d_{k,0}, M_{k;i}, M_{k;0,i})_{Ideal}$ and $\bigoplus_k (d_{k,0})_{Ideal}$ are closed under the same coproduct.

Next we define a vector space which turns out to be isomorphic to the primitive elements of the mod$-p$ cohomology of $\Omega^{2n+1}S^{2n+1}$, i.e. the dual of the generators of $H_*(\Omega^{2n+1}S^{2n+1})$. We recall that the natural product in $H^*(\Omega^{2n+1}S^{2n+1})$ is not related with the product of the Dickson algebra. Moreover, the coproduct in D is not related with the Pontryagin product in $H_*(\Omega^{2n+1}S^{2n+1})$ either.

Definition 24 *Let $C(n,p)$ be the $\mathbb{Z}/p\mathbb{Z}$-vector space given by $\bigoplus_k j_n((d_{k,0}, M_{k;i}, M_{k;0,i})_{Ideal})$.*

We are interested in finding all elements in $C(n,p)$ which are annihilated by all Steenrod operations. Those elements shall be computed in the next section.

Definition 25 *Let $(d_{k,0})_{Ideal}$ be the principal ideal in $D[k]$ and $j_n((d_{k,0})_{Ideal})$ its image in $D[k]_n$. Let $Ann(j_n(d_{k,0})_{Ideal})$ stand for the space generated by all monomials in $j_n(d_{k,0})_{Ideal}$ which are annihilated by all Steenrod operations.*

We close this section by examining the map j_n, because of its applications in the next section.

Lemma 26 *Let $d^m = d_{k,0}^{m_0}...d_{k,i}^{m_i}$ such that $m_t \equiv 0 \bmod p^{k-i}$ for $1 \le t \le i$ and $m_i \ge p^{k-i}$. Let $n = \sum(m)$. Then $j_n(d^m d_{k,k-1}) \equiv d^m d_{k,i-1}^{p^{k-i}}/d_{k,i}^{p^{k-i}}$ + (terms of polynomial degree less than $n+3-p^{k-i}$).*

Proof. Let $I = \chi_{\min}(d^m d_{k,k-1})$, then $i_k = n+1$. We are interested in admissible sequences in $\hat{\imath}(d^m d_{k,k-1}) = \prod_{t=0}^{i} \hat{\imath}(d_{k,t}^{p^{k-i}})^{\frac{m_t}{p^{k-i}}}\hat{\imath}(d_{k,k-1})$ such that they have polynomial degree less than $n+1$. Because of the restriction on the degree,

we must consider a summand of at least one element $\hat{\imath}(d_{k,t}^{p^{k-i}})$ in $\hat{\imath}(d^m d_{k,k-1})$ such that it is not divisible by h_k. That summand will be a p-th power because of formula 1. There are two distinct choices: 1) $\hat{\imath}(d_{k,k-1})$ and 2) $\hat{\imath}(d_{k,t}^{p^{k-i}})$ for $t \leq i$. 1) In order for the corresponding sequence to be admissible, we should consider $h_t^{p^{k-t}}$ in $\hat{\imath}(d_{k,k-1})$ for $t \leq i$. The smallest one will be the one with $t = i$ namely:

$$I' = I - I_{k,k-1} + (0, ..., 0, p^{k-i}, \underbrace{0, ..., 0}_{k-i})$$

Moreover, $(\theta^*)^{-1} h^{I'} = d^m d_{k,i-1}^{p^{k-i}} / d_{k,i}^{p^{k-i}}$. For $I' = I - I_{k,k-1} + (0, ..., 0, p^{k-i-t}, \underbrace{0, ..., 0}_{k-i-t})$, it is not admissible. One more case where the polynomial degree is n is described as follows: let $\prod_{s=1}^{k} h_s^{a_s p^{t_s}}$ be a summand in $\hat{\imath}(d_{k,t}^{p^{k-i}})$ with $a_{s_0} = 0$ for $k > s_0 > i$ and $h_{s_0}^{p^{k-s_0}}$ the corresponding summand from $d_{k,k-1}$. Let I' be the corresponding sequence, then $h^{I'}$ is divisible by $h_s^{p^{k-i}}$ for all s except for s_0. It follows that I' is not admissible.

2) Let $\prod_{s=1}^{k-1} h_s^{a_s p^{t_s}}$ be a summand in $\hat{\imath}(d_{k,t}^{p^{k-i}})$ and the corresponding sequence I'.Because at least $h_k^{p^{k-i}}$ has been divided from $\theta^*(d^m d_{k,k-1})$, $i'_k \leq n + 1 - (p^{k-i} - 1)$.

Up to this point, we have considered the composition $\pi_n(\Phi_k(d^m d_{k,k-1}))$ and we should take care of Φ_k^{-1}. It is obvious from case 1) and 2) above that the only possible admissible sequences depending on m are of the form

$$I'(t) = I - I_{k,k-1} + (0, ..., 0, p^{k-i+t}, \underbrace{0, ..., 0}_{k-i+t})$$

Here $t \geq 0$ and $I'(0)$ is the smallest one. But $\Phi_k(\theta^*(h^{I'(0)}))$ and $\Phi_k(d^m d_{k,k-1})$ contains $(Q_{I'(t)})^*$ with the same coefficient for $t > 0$. Finally, $j_n(d^m d_{k,k-1}) = d^m d_{k,i-1}^{p^{k-i}} / d_{k,i}^{p^{k-i}}$ + (other terms of degree less than $n + 3 - p^{k-i}$). ■

Corollary 27 *Let $d^m = d_{k,0}^{m_0} ... d_{k,i}^{m_i}$ such that $m_t \equiv 0 \bmod p^{k-i+l}$ for $1 \leq t \leq i$ and $m_i = ap^{k-i+l} + p^{k-i+l+1} b$ for $1 \leq a \leq p-1$ and $b \geq 0$. Let $n = \sum(m)$. Then $j_n(d^m d_{k,k-1}^{ap^l}) \equiv d^m d_{k,i-1}^{ap^{k-i+l}} / d_{k,i}^{ap^{k-i+l}}$ + (terms of degree less than $n + 3 - p^{k-i+l}$).*

Example 28 *1) Let $p = 3$ and $n = 22$. We recall that $j_n = \Phi_k^{-1}(\pi_n(\Phi_k))$. Then $\Phi_k^{-1}(\pi_n(\Phi_k(d_{3,0} d_{3,1}^{21} d_{3,2}^2))) = 0$.*
2) Let $n = 19$, then $\Phi_k^{-1}(\pi_n(\Phi_k(d_{3,0} d_{3,1}^{18} d_{3,2}^2))) = d_{3,0}^{19}$.

Remark 29 *Campbell, Peterson and Selick provided a convenient method for calculating j_n for relatively small n, [1].*

3 The Steenrod algebra action and $Ann(j_n(d_{k,0})_{Ideal})$

We recall some well known facts about the Steenrod algebra action on generators of $H^*((BZ/p\mathbb{Z})^k, \mathbb{Z}/p\mathbb{Z})^{GL_k}$, please see [7] for details.

Theorem 30 *[7]a)* $P^{p^j}(d_{k,i}) = \begin{cases} d_{k,i-1}, \text{ if } j = i-1 \\ -d_{k,i}d_{k,k-1}, \text{ if } j = k-1 \\ 0, \text{ otherwise} \end{cases}$

b) $P^n(h_k) = \begin{cases} \frac{1}{d_{k-1,0}}(P^n(d_{k,0}) - h_k P^n(d_{k-1,0})) \\ 0, \text{ if } C(n) = \emptyset \end{cases}$. *Here $C(n)$ stands for all sequences consisting of non-negative integers $c = [c_0, ..., c_{k-1}]$ such that*

$$n = \sum_0^{k-1} c_t(p^{k-1} + ... + p^{k-1-t})$$

c) $\beta M_{k;0} L_k^{p-2} = d_{k,0}$.
d) $\beta M_{k;0,s} L_k^{p-2} = -M_{k;s} L_k^{p-2}$, *for* $s > 0$.
e) $P^{p^{s-1}} M_{k;s} L_k^{p-2} = M_{k;s-1} L_k^{p-2}$, *for* $s > 0$.

It is immediate from the theorem above that:

Proposition 31 *1) No-element of $S(E_k)^{GL_k}$ is annihilated by all Steenrod operations.*

2) For any monomial M in $S(E_k)^{GL_k}$ there exists a Steenrod operation P^l such that M and $P^l M$ have the same polynomial degree.

In order to discuss the effect of the Steenrod algebra action on polynomial degrees we examine the action of P^q on the Dickson algebra generators.

Lemma 32 *Let the Peterson monomial $f = y_k^{p^k} \cdots y_{t+1}^{p^{t+1}} y_t^{p^{t-1}} \cdots y_1$. The Steenrod algebra action on f, $P^q f$ is not identically zero if and only if $q = \sum_{0, i \neq t}^{k} a_i p^i$ for $0 \leq a_i \leq 1$. In that case,*

$$P^q f = \prod_{1+t}^{k} y_i^{p^i(1+a_i(p-1))} \prod_1^t y_i^{p^{i-1}(1+a_{i-1}(p-1))}$$

Proposition 33 *Let the polynomials $L_k = \sum_{\sigma \in \Sigma_k} sign(\sigma) y_{\sigma(k)}^{p^{k-1}} ... y_{\sigma(1)}$ and $L_{k,t} = \sum_{\sigma \in \Sigma_k} sign(\sigma) y_{\sigma(k)}^{p^k} \cdots y_{\sigma(t+1)}^{p^{t+1}} y_{\sigma(t)}^{p^{t-1}} ... y_{\sigma(1)}$ which are the expand of the following determinants:*

$$L_k := \begin{vmatrix} y_1 & \cdots & y_k \\ \vdots & & \vdots \\ y_1^{p^{k-1}} & \cdots & y_k^{p^{k-1}} \end{vmatrix} \text{ and } L_{k,t} = \begin{vmatrix} y_1 & \cdots & y_k \\ \vdots & & \vdots \\ y_1^{p^k} & \cdots & y_k^{p^k} \end{vmatrix}$$

In the last determinant the row $(\; y_1^{p^t} \quad \cdots \quad y_k^{p^t} \;)$ *is deleted.*

1) $P^q L_k \neq 0$ *if and only if* $q = \sum_t^{k-1} p^i$ *or* 0 *and in that case and the action is given as follows:*

$$P^q L_k = \sum_{\sigma \in \Sigma_k} sign(\sigma) y_{\sigma(k)}^{p^k} \cdots y_{\sigma(t+1)}^{p^{t+1}} y_{\sigma(t)}^{p^{t-1}} \cdots y_{\sigma(1)} = L_{k,t}$$

2) $P^q L_{k,t} \neq 0$ *if and only if*

$$q = \begin{cases} \sum_s^k p^i \text{ for } s \geq t+1 \\ \sum_l^{t-1} p^i \text{ for } l \geq 0 \\ \sum_s^k p^i + \sum_l^{t-1} p^i \text{ for } s \geq t+1 \text{ and } l \geq 0 \end{cases}$$

and in that case

$$P^q L_{k,t} = \begin{cases} \sum_{\sigma \in \Sigma_k} sign(\sigma) y_{\sigma(k)}^{p^{k+1}} \cdots y_{\sigma(s)}^{p^{s+1}} y_{\sigma(s-1)}^{p^{s-1}} \cdots y_{\sigma(t+1)}^{p^{t+1}} y_{\sigma(t)}^{p^{t-1}} \cdots y_{\sigma(1)} \\ \sum_{\sigma \in \Sigma_k} sign(\sigma) y_{\sigma(k)}^{p^k} \cdots y_{\sigma(t+1)}^{p^{t+1}} y_{\sigma(t)}^{p^t} \cdots y_{\sigma(l+1)}^{p^{l+1}} y_{\sigma(l)}^{p^{l-1}} \cdots y_{\sigma(1)} \\ \sum_{\sigma \in \Sigma_k} sign(\sigma) y_{\sigma(k)}^{p^{k+1}} \cdots y_{\sigma(s)}^{p^{s+1}} y_{\sigma(s-1)}^{p^{s-1}} \cdots y_{\sigma(t+1)}^{p^{t+1}} y_{\sigma(t)}^{p^t} \cdots y_{\sigma(l+1)}^{p^{l+1}} y_{\sigma(l)}^{p^{l-1}} \cdots y_{\sigma(1)} \end{cases}$$

Proof. We prove 1) and the proof for 2) is identical. Let q and q' as in lemma 32 such that $q = q' + p^{t+s} + p^t$ with $s > 1$. Then

$$P^q f = \sum_{\sigma \in \Sigma_k} sign(\sigma) P^{q' + p^{t+s}} \left(\frac{y_{\sigma(k)}^{p^{k+1}} \cdots y_{\sigma(1)}}{y_{\sigma(t+2)}^{p^{t+1}} y_{\sigma(t+1)}^{p^t}} \right) y_{\sigma(t+2)}^{p^{t+1}} y_{\sigma(t+1)}^{p^{t+1}}$$

There exists a σ' such that $\sigma' = (t+1, t+2)\sigma$ and $sign(\sigma') = -sign(\sigma)$. Thus $P^q f = 0$. ■

Corollary 34 *1) Let* $g = f^{p^l}$ *and* $f = L_k$, $L_{k,t}$, *then* $P^q g = (P^{\frac{q}{p^l}} f)^{p^l}$.

2) Let $g = L_{k,t}$ *and* $q < p^k$, *then* $P^q g \neq 0$ *if and only if* $q = p^{t-1} + \ldots + p^l$ *for* $l \geq 0$.

Lemma 35 $\prod_t^{k-2} \binom{a_{i+1}}{a_i} = \sum_1^{k-t} \binom{a_{k-1}-1}{a_{k-2}-1} \cdots \binom{a_{k-i}-1}{a_{k-i-1}} \binom{a_{k-i-1}}{a_{k-i-2}} \cdots \binom{a_{k-(k-t-1)}}{a_{k-(k-t)}}$.

Theorem 36 *1) Let* $q = \sum_t^{k-1} a_i p^{i+l}$ *such that* $p - 1 \geq a_i \geq a_{i-1} > a_{t-1} = 0$. *Then*

$$P^q d_{k,0}^{p^l} = d_{k,0}^{p^l} (-1)^{a_{k-1}} \prod_t^{k-1} \binom{a_i}{a_{i-1}} d_{k,i}^{p^l (a_i - a_{i-1})}$$

Otherwise, $P^q d_{k,0}^{p^l} = 0$.

2) Let $q = \sum_{s}^{k-1} a_i p^{i+l} > 0$ *such that* $p-1 \geq a_i \geq a_{i-1} \geq a_t \geq 0$ *and* $a_t + 1 \geq a_{t-1} \geq a_i \geq a_{i-1} \geq 0$. *Then*

$$P^q d_{k,t}^{p^l} = d_{k,t}^{p^l} (-1)^{a_{k-1}} \left(\prod_{t+1}^{k-1} \binom{a_i}{a_{i-1}} \right) \binom{a_t+1}{a_{t-1}} \left(\prod_{s}^{t-1} \binom{a_i}{a_{i-1}} \right) \prod_{s}^{k-1} d_{k,i}^{p^l(a_i - a_{i-1})}$$

Here $a_{s-1} = 0$. *Otherwise,* $P^q d_{k,0}^{p^l} = 0$.

Remark 37 *Please note that the case* $a_t = 0$ *and* $a_{t-1} = 1$ *is allowed.*

Proof. The idea of the proof has been used in [9]. We use induction and the Cartan formula on the identity:

$$d_{k,t}^{p^l} L_k^{p^l} = L_{k,t}^{p^l} \tag{3}$$

For convenience we shall use the notation $(a_{k-1}, ..., a_t) f := P^{a_{k-1}p^{k-1+l} + ... + a_t p^{t+l}} f$ in the sequel of the proof.

1) We prove the claimed formula for $(1, ..., 1) d_{k,0}^{p^l}$ using formula 3, proposition 33 1) and corollary 34 1). Using induction hypothesis, formula 3 and lemma 35 the formula follows.

2) If $a_{t-1} = 0$, then the proof is identical as in 1). If $a_t = 0$ and $a_{t-1} > 0$, then we use formula 3, proposition 33 1) and 2) and corollary 34. Let $a_t a_{t-1} > 0$. We prove the claimed formula for the following sequences using formula 3, proposition 33 and corollary 34:

$$\left(\underbrace{1, ..., 1}_{k-t}, \underbrace{1, ..., 1}_{t-s} \right), \left(\underbrace{1, ..., 1}_{k-t}, \underbrace{2, ..., 2}_{t-s} \right), \left(\underbrace{1, ..., 1}_{k-t}, \underbrace{2, ..., 2, 1..., 1}_{t-s} \right)$$

Using induction hypothesis, formula 3 and lemma 35 the formula follows. ■

The following Theorem is an application of last Theorem and the Cartan formula.

Theorem 38 *Let* $d = d_{k,0}^{ap^l} d_{k,i}^{ap^{k-i+l}}$ *for* $1 \leq a \leq p-1$. *Then* $P^q d$:

$$P^q d = \begin{cases} \sum_{c=0}^{b} (-1)^c \binom{a}{c} \binom{a}{b-c} d_{k,0}^{ap^l} d_{k,k-t}^{cp^l} d_{k,i-t}^{(b-c)p^{k-i+l}} d_{k,i}^{(a-b+c)p^{k-i+l}} \\ \qquad \text{for } q = b(p^{k-1+l} + ... + p^{k-t+l}), b \leq 2a. \\ (-1)^b \binom{a}{b} d_{k,0}^{ap^l} d_{k,k-1}^{bp^l} d_{k,i}^{ap^{k-i+l}} \quad \text{for } q = bp^{2k-1-i+l}, b \leq a. \\ (-1)^c \binom{a}{c} d_{k,k-1}^{cp^{k-i+l}} d_{k,i}^{cp^{k-i+l}} P^{bp^{k-1+l}} (d_{k,0}^{ap^l} d_{k,i}^{(a-c)p^{k-i+l}}) \\ \qquad \text{for } q = bp^{k-1+l} + cp^{2k-1-i+l}, \ b, c \leq a. \end{cases}$$

Proof. We use Theorem 30 and Cartan formula. ■

Corollary 39 *Let* $d = d_{k,0}^{ap^l} d_{k,i}^{ap^{k-i+l}}$ *for* $1 \le a \le p-1$ *and* $n = a(p^l + p^{k-i+l})$. *Then* $j_n(P^q d)$ *is non-zero for* $a > 1$, $q = b(p^{k-1+l} + ... + p^{k-t+l})$ *and* $b = 2$. *Moreover,* $j_n(P^q d) = 0$ *for all* $q \ge 0$ *if and only if* $a = 1$.

Example 40 *Let* $p = 3$, $n = 2(1 + p^{k-i})$ *and* $d = d_{k,0}^2 d_{k,i}^{2p^{k-i}}$. *Let* $q = 2p^{k-1}$, *then* $P^q d = d_{k,0}^2 d_{k,k-1}^2 d_{k,i}^{2p^{k-i}} + d_{k,0}^2 d_{k,i-1}^{2p^{k-i}} - 4d_{k,0}^2 d_{k,k-1} d_{k,i-1}^{p^{k-i}} d_{k,i}^{p^{k-i}}$ *and* $j_n(P^q d) = d_{k,0}^2 d_{k,i-1}^{2p^{k-i}}$.

Theorem 41 *Let* $m = (m_0, ..., m_{k-1})$ *and* $d^m = \prod d_{k,i}^{m_i}$ *such that* $\sum m_i = n$.

1) Let $m_0 = n$, *then* $d_{k,0}^n \in Ann(j_n(d_{k,0})_{Ideal})$.

2) Let $m_0 = \sum_{0 \le t \le l(0)} m_{0,s_t} p^{s_t} < n$ *and* $m_i = 0$ *for* $1 \le i \le k-1$. *then* $d_{k,0}^n \in Ann(j_n(d_{k,0})_{Ideal})$ *if and only if* $m_0 + p^{s_0 - 1} \le n < m_0 + p^{s_0}$.

3) Let $m_0 < n$ *and* $\exists i$ *such that* $m_i \ne 0 \bmod p$, *then* $d^m \notin Ann(j_n(d_{k,0})_{Ideal})$.

4) Let $m_0 = \sum_{0 \le t \le l(0)} m_{0,s_t} p^{s_t}$ *with* $\prod_{0 \le t \le l(0)} m_{0,s_t} \ne 0$. $d^m \in Ann(j_n(d_{k,0})_{Ideal})$ *if and only if* $m_{0,s_t} = 1$ *for all* t *and* $d^m = \prod_{0 \le t \le l(0)} d_{k,0}^{m_{0,s_t} p^{s_t}} d_{k,i_{s_t}}^{m_{0,s_t} p^{k-i_{s_t}+s_t}}$.

Proof. 1) It follows from Theorem 30 that $P^a(d_{k,0}^n) \in (d_{k,0}^n)_{ideal}$. Moreover, if $a > 0$ and $P^a(d_{k,0}^n) \ne 0$, then its polynomial degree is greater than n. Thus $d_{k,0}^n \in Ann(j_n(d_{k,0})_{Ideal})$.

2) This case is similar to the previous one taking on to account binomial coefficients as well. If n is outside the given range, then $P^{ap^{k-1}} d_{k,0}^{m_0} = m_{0,s_0} d_{k,0}^{m_0} d_{k,k-1}^a$ for $a = p^{s_0}$.

3) Let i be minimal with this property, then $P^{p^{i-1}} d^m = m_{i,0} d^m d_{k,i-1}/d_{k,i}$.

4) Let $\mu(m_i)$ such that $m_i \equiv 0 \bmod p^{\mu(m_i)}$ and $m_i \not\equiv 0 \bmod p^{\mu(m_i)+1}$. Let $T(m, \min) = \{i | i > 0, i - 1 + \mu(m_i) = \min\{t - 1 + \mu(m_t) | t > 0\}\}$.
Let $t_0 \in T(m, \min)$ and $t_0 - 1 + \mu(m_{t_0}) < k - 1 + \mu(m_0)$, then using Theorem 30 we get $j_n(P^{p^{t_0-1+\mu(m_{t_0})}} d^m) \ne 0$.
Let $t_0 = \max T(m, \min)$ and $t_0 - 1 + \mu(m_{t_0}) > k - 1 + \mu(m_0)$, then using Theorem 30 and corollary 27 we get $j_n(P^{p^{k-1+\mu(m_0)}} d^m) = -m_{0,\mu(m_0)} d^m d_{k,t_0-1}^{p^{k-t_0+\mu(m_0)}} / d_{k,t_0}^{p^{k-t_0+\mu(m_0)}}$.
Here we also use $k - t_0 + \mu(m_0) < \mu(m_{t_0})$.
Let $t_0 = \max T(m, \min)$ and $t_0 - 1 + \mu(m_{t_0}) = k - 1 + \mu(m_0)$, then using Theorem 30 and corollary 27 we get $j_n(P^{p^{k-1+\mu(m_0)}} d^m) = (m_{t_0,\mu(m_{t_0})} - m_{0,\mu(m_0)}) d^m d_{k,t_0-1}^{p^{k-t_0+\mu(m_{t_0})}} / d_{k,t_0}^{p^{k-t_0+\mu(m_{t_0})}} + \sum_{t_0' \in T(m,\min)-\{t_0\}} m_{t_0',\mu(m_{t_0'})} d^m d_{k,t_0'-1}^{p^{k-t_0'+\mu(m_{t_0'})}} / d_{k,t_0'}^{p^{k-t_0'+\mu(m_{t_0'})}}$. The last equation is zero only if $m_{t_0,\mu(m_{t_0})} = m_{0,\mu(m_0)}$ and $T(m, \min) = \{t_0\}$.
Thus if $d^m \in Ann(j_n(d_{k,0})_{Ideal})$, then d^m is divisible by $d_{k,0}^{m_{0,\mu(m_0)} p^{\mu(m_0)}} d_{k,t_0}^{m_{0,\mu(m_0)} p^{k-t_0+\mu(m_0)}}$.

Following the same method for the next non-zero coefficient, m_{0,s_1}, we deduce that if $d^m \in Ann(j_n(d_{k,0})_{Ideal})$, then the following hypothesis holds:
i) $m_0 = m_0' + m_{0s_1} p^{s_1} + m_{0,\mu(m_0)} p^{\mu(m_0)}$,

ii) $\exists! t_0$ such that $m_i \equiv 0 \bmod p^{\mu(m_{t_0})-t}$ for $1 \leq t \leq \mu(m_{t_0})-1$ for all i and $m_{t_0} \equiv m_{t_0,\mu(m_{t_0})} \bmod p^{\mu(m_{t_0})}$, $m_i \equiv 0 \bmod p^{\mu(m_{t_0})}$, $i \neq t_0$ and $m_{t_0,\mu(m_{t_0})} = m_{0,\mu(m_0)}$.
iii) $\exists! t_1$ such that $m_i \equiv 0 \bmod p^{\mu(m_{t_0})+t}$ for $1 \leq t \leq s_{t_1} - 1$ for all $i \neq t_0$, $m_{t_0} - m_{t_0,\mu(m_{t_0})} p^{\mu(m_{t_0})} \equiv 0 \bmod p^{\mu(m_{t_0})+t}$, $m_{t_1} \equiv m_{t_1,s_{t_1}} \bmod p^{p^{s_{t_1}}}$, $m_i \equiv 0 \bmod p^{s_{t_1}}$, $i \neq t_0, t_1$, $m_{t_0} - m_{t_0,\mu(m_{t_0})} p^{\mu(m_{t_0})} \equiv 0 \bmod p^{s_{t_1}}$ and $m_{t_1,s_{t_1}} = m_{0,s_1}$.

Finally, applying the same procedure for all non-zero coefficients of m_0 in its p-adic expansion we prove that: if $d^m \in Ann(j_n(d_{k,0})_{Ideal})$, then d^m is divisible by $d_{k,0}^{m_{0,s_t} p^{s_t}} d_{k,i_{s_t}}^{m_{0,s_t} p^{k-i_{s_t}+s_t}}$ for all m_{0,s_t} in m_0.

For the other direction we use induction on the number of non-zero coefficients of m_0, Cartan formula, and corollary 39. ■

Example 42 *1) Let $n = 1$. Then*
$j_1((d_{k,0}, M_{k;i}, M_{k;0,i})_{Ideal}) = \{d_{k,0}, M_{k;i}, M_{k;0,t} |\ 0 \leq i \leq k-1.1 \leq t \leq k-1\}$ *and* $Ann(j_1(d_{k,0})_{Ideal}) = \{d_{k,0}\}$.

2) Let $n = 2$. Then
$j_2((d_{k,0}, M_{k;i}, M_{k;0,i})_{Ideal}) = \{d_{k,0}d_{k,s}, d_{k,s}M_{k;i}, d_{k,s}M_{k;0,t}, M_{k;i}M_{k;s,t}, d_{k,0}M_{k;s,t} |$ $0 \leq i, s, q \leq k-1., 1 \leq t \leq k-1\}$ *and* $Ann(j_2(d_{k,0})_{Ideal}) = \{d_{k,o}^2\}$.

3) Let $n = p+1$. Then $j_n((d_{k,0})_{Ideal}) = \{d_{k,0}^{a_0} d_{k,i_1}^{a_{i_1}} ... d_{k,i_l}^{a_{i_l}} |\ 0 \leq a_t \leq p, 0 < a_0, \sum a_t \leq n\}$ *and* $Ann(j_n(d_{k,0})_{Ideal}) = \{d_{k,o}^n, d_{k,o}^p, d_{k,o}d_{k,k-1}^p\}$.

4 $H_*(\Omega^{m+1}S^{m+1};\mathbb{Z}/p\mathbb{Z})$ and Dyer-Lashof operations

We recall the description of $H_*(\Omega^{m+1}S^{m+1};\mathbb{Z}/p\mathbb{Z})$ in terms of lower Dyer-Lashof operations from [2] and [1].

Definition 43 *Let $R_n =< Q_{(I,\varepsilon)} | (I,\varepsilon) \in <\mathbb{N}, \frac{1}{2}>^k \times (\mathbb{Z}/2Z)^k$ with $i_k \leq n$ and admissible as in definition 8 for $k \geq 0 >$.Let $Q_0R_n =< Q_{(I,\varepsilon)} \in R_n$ with $i_1 = 0$ >.*

$H_*(\Omega^{m+1}S^{m+1};\mathbb{Z}/p\mathbb{Z})$ is the free commutative graded algebra on R_n/Q_0R_n.

We recall that R_n/Q_0R_n is an opposite Steenrod algebra A_* module through Nishida relations. Let consider all elements in R_n with fixed length k:

$$R_n[k] =< Q_{(I,\varepsilon)} \in R_n, (I,\varepsilon) \in < \mathbb{N}, \frac{1}{2} >^k \times (\mathbb{Z}/2Z)^k >$$

and $Q_0R_n[k] =< Q_{(I,\varepsilon)} \in R_n[k] \cap Q_0R_n,\ (I,\varepsilon) >$. Those are A_* coalgebras.

Dualizing, we get a Steenrod algebra module isomorphism:

$$PH^*(\Omega^{m+1}S^{m+1};\mathbb{Z}/p\mathbb{Z}) \approxeq \bigoplus_k (R_n[k]/Q_0R_n[k])^*$$

Next we recall the relation between $R[k]$ and modular invariants. First we note that the hom dual of $R[k]$, $R[k]^*$, coincides with what we defined in 11.

Theorem 44 *[6] $R[k]^* \approxeq D[k] \otimes S(E_k)^{GL_k}$ as algebras over the Steenrod algebra.*

Using Theorem 1.7 in [1], we deduce the following Theorem.

Theorem 45 *1)* $(R[k]/Q_0R[k])^* \approxeq ((d_{k,0}, M_{k;i}, M_{k;0,i})_{Ideal})$.
2) $PH^*(\Omega_0^{2n+1}S^{2n+1};\mathbb{Z}/p\mathbb{Z}) \approxeq j_n((d_{k,0}, M_{k;i}, M_{k;0,i})_{Ideal})$.

We quote Campbell, Peterson and Selick's Theorem 2.5 from [1].

Theorem 46 *[1]* $Ann(PH^*(\Omega_0^{2n+1}S^{2n+1};\mathbb{Z}/p\mathbb{Z}))$ *is generated by* $\{(Q_I)^* |\ I$ *admissible and satisfies* $i_s + p^t \leq n \Rightarrow 2i_s + |I_{s-1}| \equiv 0 \bmod p^{t+1}$ *for all* $t \geq 0$ *and* $1 \leq s \leq k\}$.

Our Theorem 41 describes the isomorphic image of $Ann(PH^*(\Omega_0^{2n+1}S^{2n+1};\mathbb{Z}/p\mathbb{Z}))$ in terms of Dickson invariants. It is not obvious how to decompose an element of the hom-dual basis of $R[k]^*$ with respect to the monomial basis of $D[k] \otimes S(E_k)^{GL_k}$. We describe an algorithm regarding this passage in [8].

References

[1] Campbell H. E. A., Peterson F. P. and Selick P. S.: *Self-maps of loop spaces. I* . Trans. A. M. S. **293** no1 (1986),1-39.

[2] Cohen F., Lada T., and May P.J.: **The homology of iterated loop spaces**. Lecture notes in Mathematics, **533**, (1975).

[3] Dickson L.E.: *A fundamental system of invariants of the general modular linear group with a solution of the form problem.* Trans. Amer. Math. Soc., **12**, 75-98, (1911).

[4] Huyhn Mui: *Modular invariant theory and the cohomology algebras of the symmetric groups*, J. Fac. Sci. Univ. Tokyo, **IA** (1975), 319-369.

[5] Huynh Mui: *Homology operations derived from modular coinvariants.* Lecture Notes in Mathematics, **172**, 85-115, (1985).

[6] Kechagias E. Nondas: *Extended Dyer-Lashof algebras and modular coinvariants.* Manuscripta Math. **84**, 1994, 261-290.

[7] Kechagias, E. Nondas: *The Steenrod algebra action on generators of rings of invariants of subgroups of* $GL_n(\mathbb{Z}/p\mathbb{Z})$. Proc. Amer. Math. Soc. **118**, 943-952, (1993).

[8] Kechagias, E. Nondas.: *Adem relations in the Dyer-Lashof algebra and modular invariants.* Preprint, 2002, 30 pages.

[9] Kechagias, E. Nondas.: *Homology operations and modular invariant theory.* Ph. D. Thesis, Queen's University, 1990.

[10] Madsen, I.: *On the action of Dyer-Lashof algebra in* $H_*(B)$. Pacific Journal of Mathematics, **60**, 235-275, (1975).